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EXTREMUM PRINCIPLES FOR CRACK ENERGY-RELEASE IN ELASTIC BODY

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Abstract—The minimum principle and the maximum principle, complementary to each other, are established in order to evaluate the energy-release corresponding to prescribed crack shape in the elastic body. Frictionless bounds of the crack surfaces are considered. A new approach to the problem proposed in the study is based on methods of convex analysis. A brief introduction to the concepts of convex analysis used in the study is presented in the appendix.

1. INTRODUCTION

An original construction of the extremum principles, complementary to each other, corresponding to the energy-release problem for the elastic-perfectly plastic body was proposed in [1] and developed in [2, 3]. The principles make it possible to calculate lower and upper bounds for the energy released from the body subjected to prescribed change of the material properties while the external loading remains unchanged.

The principles have been used in [2] to solve a crack propagation problem for the elastic body. More precisely, the principles were applied in [2] to solve the auxiliary problem of three-dimensional zero-strength inclusion within the body. The final result was obtained as a limit resulting from the shrinkage of the inclusion to obtain the considered two-dimensional crack.

The above approach, justified for the particular plane crack considered in [2], is not satisfactory to analyse a general crack problem for two reasons. The first reason is that the shrinkage of the inclusion leads to large deformations in its interior, what violates the assumption of small deformations. The second reason is that one can not exclude in advance the interaction between the crack surfaces.

In the present work a new approach to a general crack problem for the elastic body is proposed. Instead of introducing an artificial zero-stress inclusion we model the crack using the concepts of convex analysis. Namely, we introduce lower-semicontinuous convex function $G(\omega)$ which represents the frictionless bounds for the crack surfaces. The argument ω represents the relative normal displacements of the crack surfaces and the parameter τ, dual to ω, represents the normal tractions on these surfaces.

The proposed approach provides a uniform mathematical formulation and solution of the problem. Namely, both the constitutive relations and the internal boundary conditions are described with the concept of the subgradient. Consequently, in order to derive the extremum principles, we use throughout the article, the concepts and methods of convex analysis. The basic concepts are presented in Section 2. The problem is formulated in Section 3. The necessary derivations are given in Section 4. The resulting extremum principles are established in Section 5.

2. BASIC NOTIONS

The behavior of generalized elastic body is here determined by the *free energy function* $W(\boldsymbol{\epsilon}, \mathbf{x})$ which represents the material properties of nonhomogeneous body V. The function $W(\boldsymbol{\epsilon}, \mathbf{x})$ defined for all symmetric *strain tensors* $\boldsymbol{\epsilon}$ and all $\mathbf{x}$ from V is assumed to be *lower-semicontinuous* and *convex*[4, 5] with respect to $\boldsymbol{\epsilon}$ and sufficiently regular with respect to $\mathbf{x}$ (to assure the existence of the volume integrals appearing in the sequel). Moreover it is assumed that $W(\boldsymbol{\epsilon}, \mathbf{x})$ attains the minimum equal to zero at $\boldsymbol{\epsilon} = \mathbf{0}$ for arbitrary $\mathbf{x}$ from V.

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The *constitutive relation* for the elastic body takes the form

$$\sigma(\mathbf{x}) \in \partial W(\boldsymbol{\epsilon}, \mathbf{x})\,|_{\boldsymbol{\epsilon}=\boldsymbol{\epsilon}(x)} \qquad \mathbf{x} \in V \tag{1}$$

where $\sigma(\mathbf{x})$ is the stress function corresponding to the strain function $\boldsymbol{\epsilon}(\mathbf{x})$ and $\partial W(\boldsymbol{\epsilon}, \mathbf{x})$ is the *subdifferential*[4, 5] of W with respect to $\boldsymbol{\epsilon}$.

It should be noted that the relation (1) does not imply one-to-one correspondence between the strain and stress tensors. In the particular case when function $W(\boldsymbol{\epsilon}, \mathbf{x})$ is differentiable with respect to $\boldsymbol{\epsilon}$ the subdifferential contains exactly one element identified with the *gradient* of W with respect to $\boldsymbol{\epsilon}$. Then the constitutive relation (1) takes the classical form

$$\sigma(\mathbf{x}) = \frac{\partial W(\boldsymbol{\epsilon}, \mathbf{x})}{\partial \boldsymbol{\epsilon}}\bigg|_{\boldsymbol{\epsilon}=\boldsymbol{\epsilon}(x)} \qquad \mathbf{x} \in V. \tag{2}$$

In order to derive the extremum principles we shall also use the concept of the *complementary energy* $W^*(\sigma, \mathbf{x})$ defined for every symmetric stress tensor σ and all $\mathbf{x}$ from V as the function *conjugate*[4] or *polar*[5] to $W(\boldsymbol{\epsilon}, \mathbf{x})$

$$W^*(\sigma, \mathbf{x}) = \sup_{\boldsymbol{\epsilon}}\,[\boldsymbol{\epsilon}\cdot\sigma - W(\boldsymbol{\epsilon}, \mathbf{x})] \qquad \mathbf{x} \in V \tag{3}$$

where the dot denotes the scalar product.

The same concepts of convex analysis are also used in the present work to describe the internal boundary conditions on the surfaces of the crack. Namely, lower-semi-continuous convex function $G(\omega)$ introduced in Section 3 determines the frictionless bounds on the internal surface B_0. The complementary function $G^*(\tau)$ conjugate[4] or polar[5] to $G(\omega)$ is defined by

$$G^*(\tau) = \sup_{\omega}\,[\omega\,\tau - G(\omega)] \tag{4}$$

where ω is the measure of relative normal displacements of the crack surfaces and τ is the measure of normal tractions on these surfaces.

3. FORMULATION OF THE PROBLEM

We consider two configurations of the elastic body occupying three-dimensional region V bounded by two-dimensional sufficiently regular surface B. The boundary B is decomposed into the surface B_S, where the tractions $\mathbf{T}^B(\mathbf{x})$ are prescribed, and the surface B_K, where the displacements $\mathbf{u}^B(\mathbf{x})$ are prescribed. The material properties for both configurations are determined by the free energy function $W(\boldsymbol{\epsilon}, \mathbf{x})$ defined for all strain tensors $\boldsymbol{\epsilon}$ and for all $\mathbf{x}$ from V (see Section 2).

The first configuration resulting from continuous deformation of the region V will be referred to as body 0. The second configuration resulting from continuous deformation of the region $V_1 = V\backslash C$ will be referred to as body 1. Here two-dimensional sufficiently regular region C contained in V represents the crack.

In order to describe the complete boundary conditions for body 1 we introduce closed surface B_0 determined by region C (see Fig. 1) which will be referred to as the *internal boundary* of the region V_1. The internal boundary B_0 is composed of two surfaces B_0^+ and B_0^- which have the shape of region C and are oriented with the normal unit vectors $\mathbf{n}^+$ and $\mathbf{n}^-$ in the directions opposite to each other.

Let the vector functions $\mathbf{u}^+(\mathbf{x})$ and $\mathbf{u}^-(\mathbf{x})$ defined on C denote the displacements of the surfaces B_0^+ and B_0^-, respectively. The scalar function $\omega(\mathbf{x})$ defined on C by

$$\omega(\mathbf{x}) = [\mathbf{u}^+(\mathbf{x}) - \mathbf{u}^-(\mathbf{x})]\cdot\mathbf{n}^+(\mathbf{x}), \qquad \mathbf{x} \in C \tag{5}$$

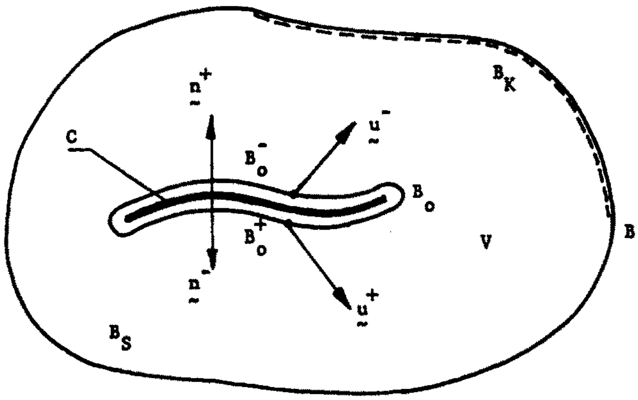

Fig. 1. Internal boundary B_0 determined by the crack region C. In order to distinguish between surfaces B_0^+ and B_0^- in the drawing the closed boundary B_0 of two-dimensional region C is translated from its actual position towards the interior of the body.

determines the normal component of the relative displacements of the crack surfaces. Neglecting the friction on the internal boundary B_0 we can describe the *internal boundary conditions* in the form

$$\tau(\mathbf{x}) \in \partial G(\omega)\,|_{\omega=\omega(\mathbf{x})}, \qquad \mathbf{x} \in C \tag{6}$$

where the function $G(\omega)$ representing the frictionless bounds is defined by

$$G(\omega) = \begin{cases} 0 & \text{if } \omega \leq 0 \\ +\infty & \text{if } \omega > 0, \end{cases} \tag{7}$$

the scalar function $\tau(\mathbf{x})$ defined on C represents the normal traction on the crack surfaces

$$\mathbf{T}(\mathbf{x}) = \begin{cases} -\tau(\mathbf{x})\,\mathbf{n}^+(\mathbf{x}) & \text{if } \mathbf{x} \in B_0^+ \\ -\tau(\mathbf{x})\,\mathbf{n}^-(\mathbf{x}) & \text{if } \mathbf{x} \times B_0^- \end{cases} \tag{8}$$

and $\partial G(\omega)$ is the subdifferential[4, 5] of the function G.

The problem consists in finding the *increment of energy*

$$\Delta E = \int_{V_1} W(\boldsymbol{\epsilon}^1, \mathbf{x})\,dV - \int_V W(\boldsymbol{\epsilon}^0, \mathbf{x})\,dV - \int_{B_S} \mathbf{T}^B \cdot (\mathbf{u}^1 - \mathbf{u}^0)\,dB \tag{9}$$

corresponding to the transition from body 0 to body 1. Here the actual strains in body 0 are represented by the function $\boldsymbol{\epsilon}^0(\mathbf{x})$ defined in V, the actual strains in body 1 are represented by the function $\boldsymbol{\epsilon}^1(\mathbf{x})$ defined in V_1. The functions $\mathbf{u}^0(\mathbf{x})$ and $\mathbf{u}^1(\mathbf{x})$ are the corresponding displacements. The first integral of (9) expresses the energy stored in body 1, the second integral—the energy stored in body 0 and the third integral—the energy supplied from outside during the transition from body 0 to body 1 (provided that the boundary conditions on B remain unchanged along the transition path).

The actual strain and stress functions in body 0 satisfy the *constitutive relation*

$$\boldsymbol{\sigma}^0(\mathbf{x}) \in \partial W(\boldsymbol{\epsilon}, \mathbf{x})\,|_{\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^0(x)}, \qquad \mathbf{x} \in V \tag{10}$$

the *kinematic conditions*

$$\epsilon_{ij}^0 = \tfrac{1}{2}\,(u_{i,j}^0 + u_{j,i}^0) \text{ in } V \quad \text{and} \quad u_i^0 = u_i^B \text{ on } B_K \tag{11}$$

and the *static conditions*

$$\sigma^0_{ij,j} = 0 \text{ in } V \quad \text{and} \quad T^0_i \equiv \sigma^0_{ij}\, n_j = T^B_i \text{ on } B_S \tag{12}$$

where $\mathbf{n}$ denotes the unit vector normal to the boundary B.

The actual strain and stress functions in body 1 satisfy the *constitutive relation*

$$\boldsymbol{\sigma}^1(\mathbf{x}) \in \partial W(\boldsymbol{\epsilon}, \mathbf{x})\,|_{\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^1(x)}, \qquad \mathbf{x} \in V_1 \tag{13}$$

the *kinematic conditions*

$$\epsilon^1_{ij} = \tfrac{1}{2}\,(u^1_{i,j} + u^1_{j,i}) \text{ in } V_1 \quad \text{and} \quad u^1_i = u^B_i \text{ on } B_K \tag{14}$$

the *static conditions*

$$\sigma^1_{ij,j} = 0 \text{ in } V_1 \quad \text{and} \quad T^1_i \equiv \sigma^1_{ij}\, n_j = T^B_i \text{ on } B_S \tag{15}$$

and the *internal boundary conditions*

$$\tau^1(\mathbf{x}) \in \partial G(\omega)\,|_{\omega=\omega^1(x)}, \qquad \mathbf{x} \in C \tag{16}$$

where

$$\omega^1 = (u^{1+}_i - u^{1-}_i)\, n^+_i \text{ on } C \tag{17}$$

$$\sigma^1_{ij}\, n^+_j = -\tau^1\, n^+_i \text{ on } B^+_0 \quad \text{and} \quad \sigma^1_{ij}\, n^-_j = -\tau^1\, n^-_i \text{ on } B^-_0 . \tag{18}$$

4. DERIVATIONS

We introduce the following sets of the strain and stress functions:

— The set K_0 of all strain functions $\boldsymbol{\epsilon}(\mathbf{x})$ defined in V which can be derived from a displacement function $\mathbf{u}(\mathbf{x})$ $\{\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } V\}$ satisfying the prescribed boundary conditions $\{u_i = u^B_i \text{ on } B_K\}$.

— The set K_1 of all strain functions $\boldsymbol{\epsilon}(\mathbf{x})$ defined in V_1 which can be derived from a displacement function $\mathbf{u}(\mathbf{x})$ $\{\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } V_1\}$ satisfying the prescribed boundary conditions $\{u_i = u^B_i \text{ on } B_K\}$.

— The set S_0 of all stress functions $\boldsymbol{\sigma}(\mathbf{x})$ defined in V which satisfy the equilibrium equation $\{\sigma_{ij,j} = 0 \text{ in } V\}$ and the prescribed boundary conditions $\{\sigma_{ij}\, n_j = T^B_i \text{ on } B_S\}$.

— The set S_1 of all stress functions $\boldsymbol{\sigma}(\mathbf{x})$ defined in V_1 which satisfy the equilibrium equation $\{\sigma_{ij,j} = 0 \text{ in } V_1\}$, the prescribed boundary conditions $\{\sigma_{ij}\, n_j = T^B_i \text{ on } B_S\}$ and admit only normal tractions on the internal boundary in the form $\{\sigma_{ij}\, n^+_j = -\tau\, n^+_i$ on B^+_0 and $\sigma_{ij}\, n^-_j = -\tau\, n^-_i \text{ on } B^-_0\}$ where $\tau(\mathbf{x})$ is arbitrary scalar function defined on C.

In order to derive the basic inequalities we introduce the following integrals:

$$J(\boldsymbol{\epsilon}, \boldsymbol{\sigma}) = \int_{V_1} [W(\boldsymbol{\epsilon}, \mathbf{x}) - \boldsymbol{\epsilon} \cdot \boldsymbol{\sigma} + W^*(\boldsymbol{\sigma}, \mathbf{x})]\, \mathrm{d}V \tag{19}$$

$$I(\boldsymbol{\epsilon}) = \int_V [W(\boldsymbol{\epsilon}, \mathbf{x}) - W(\boldsymbol{\epsilon}^0, \mathbf{x}) - (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^0) \cdot \boldsymbol{\sigma}^0]\, \mathrm{d}V \tag{29}$$

$$I^*(\boldsymbol{\sigma}) = \int_V [W^*(\boldsymbol{\sigma}, \mathbf{x}) - W^*(\boldsymbol{\sigma}^0, \mathbf{x}) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}^0) \cdot \boldsymbol{\epsilon}^0]\, \mathrm{d}V \tag{21}$$

where $\epsilon(x)$ and $\sigma(x)$ are arbitrary strain and stress functions defined in V and the integral

$$L(\omega, \tau) = \int_C [G(\omega) - \omega\, \tau + G^*(\tau)]\, \mathrm{d}C \tag{22}$$

where $\omega(\mathbf{x})$ and $\tau(\mathbf{x})$ are arbitrary scalar functions defined on C.

It follows from the definition of the complementary functions (3) and (4) that the integrals (19) and (22) are non-negative. The constitutive relation (10) implies that the integrals (20) and (21) are also non-negative.

Substituting the equations

$$W(\epsilon^1, x) = \epsilon^1 \cdot \sigma^1 - W^*(\sigma^1, x), \quad x \in V_1 \tag{23}$$

$$W(\epsilon^0, x) = \epsilon^0 \cdot \sigma^0 - W^*(\sigma^0, x), \quad x \in V, \tag{24}$$

which follow from the constitutive relations (10) and (13), into (9) we can express the increment of energy in the form

$$\begin{aligned}\Delta E = &\int_{V_1}[W(\epsilon, x) - \epsilon \cdot \sigma + W^*(\sigma, x)]\,dV + \int_C [G(\omega) + G^*(\tau^1)]\,dC \\ &- J(\epsilon, \sigma^1) - I^*(\sigma) - L(\omega, \tau^1) \\ &- \int_{B_S} T^B \cdot (u^1 - u^0)\,dB + \int_{V_1}(\epsilon^1 - \epsilon)\cdot\sigma^1\,dV + \int_V(\epsilon - \epsilon^0)\cdot\sigma\,dV - \int_C \omega\tau^1\,dC\end{aligned} \tag{25}$$

where $\epsilon(x)$ is arbitrary function from the set K_1, $\omega(x)$ is the corresponding (5) scalar function defined on C, $\sigma(x)$ is arbitrary function from the set S_0.

For such functions $\epsilon(x)$ and $\sigma(x)$ the part of the expression (25) consisting of four last integrals vanishes. Indeed, taking into account the identity

$$\int_{B_0} u \cdot T^1\,dB = -\int_C \omega\,\tau^1\,dC \tag{26}$$

and the properties of the functions from sets K_1 and S_0 and making use of the divergence theorem we obtain

$$\begin{aligned}-\int_{B_S} T^B \cdot (u^1 - u^0)\,dB + \int_{V_1}(\epsilon^1 - \epsilon)\cdot\sigma^1\,dV \\ + \int_V(\epsilon - \epsilon^0)\cdot\sigma\,dV - \int_C \omega\,\tau^1\,dC = \int_{B_0} u^1 \cdot T^1\,dB.\end{aligned} \tag{27}$$

The internal boundary conditions (16) imply that $u^1(x)\cdot T^1(x) = 0$ on B_0 and that $G^*(\tau^1) = 0$ on C. Hence we can present eqn (25) in the form

$$\begin{aligned}\Delta E = \int_{V_1}[W(\epsilon, x) - \epsilon \cdot \sigma + W^*(\sigma, x)]\,dV + \int_C G(\omega)\,dC \\ - J(\epsilon, \sigma^1) - I^*(\sigma) - L(\omega, \tau^1).\end{aligned} \tag{28}$$

On the other hand the increment of energy (9) can be presented in the form

$$\begin{aligned}\Delta E = &-\int_{V_1}[W(\epsilon, x) - \epsilon \cdot \sigma + W^*(\sigma, x)]\,dV - \int_C [G(\omega^1) + G^*(\tau)]\,dC \\ &+ J(\epsilon^1, \sigma) + I(\epsilon) + L(\omega^1, \tau) \\ &- \int_{B_S} T^B \cdot (u^1 - u^0)\,dB + \int_{V_1}(\epsilon^1 - \epsilon)\cdot\sigma\,dV + \int_V(\epsilon - \epsilon^0)\cdot\sigma^0\,dV + \int_C \omega^1\tau\,dC\end{aligned} \tag{29}$$

where $\epsilon(x)$ is arbitrary function from the set K_0, $\sigma(x)$ is arbitrary function from the set S_1 and $\tau(x)$ is the corresponding (8) scalar function defined on C.

For such functions $\boldsymbol{\epsilon}(\mathbf{x})$ and $\boldsymbol{\sigma}(\mathbf{x})$ the part of the expression (29) consisting of four last integrals vanishes. Indeed, taking into account the identity

$$\int_{B_0} \mathbf{u}^1 \cdot \mathbf{T}\, \mathrm{d}B = -\int_C \omega^1 \tau\, \mathrm{d}C \tag{30}$$

and the properties of functions from the sets K_0 and S_1 and making use of the divergence theorem we obtain

$$-\int_{B_S} \mathbf{T}^B \cdot (\mathbf{u}^1 - \mathbf{u}^0)\, \mathrm{d}B + \int_{V_1} (\boldsymbol{\epsilon}^1 - \boldsymbol{\epsilon}) \cdot \boldsymbol{\sigma}^1\, \mathrm{d}V + \int_V (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^0) \cdot \boldsymbol{\sigma}\, \mathrm{d}V - \int_C \omega \tau^1\, \mathrm{d}C = -\int_{B_0} \mathbf{u} \cdot \mathbf{T}\, \mathrm{d}B. \tag{31}$$

The internal boundary conditions (16) imply that $G(\omega^1) = 0$ on C. Since $\mathbf{u}(\mathbf{x})$ is continuous in the region V we have

$$\int_{B_0} \mathbf{u} \cdot \mathbf{T}\, \mathrm{d}B = 0 \tag{32}$$

for arbitrary $\boldsymbol{\sigma}(\mathbf{x})$ from S_1. Hence we can present eqn (29) in the form

$$\Delta E = -\int_{V_1} [W(\boldsymbol{\epsilon}, \mathbf{x}) - \boldsymbol{\epsilon} \cdot \boldsymbol{\sigma} + W^*(\boldsymbol{\sigma}, \mathbf{x})]\, \mathrm{d}V - \int_C G^*(\tau)\, \mathrm{d}C + J(\boldsymbol{\epsilon}^1, \boldsymbol{\sigma}) + I(\boldsymbol{\epsilon}) + L(\omega^1, \tau). \tag{33}$$

5. EXTREMUM PRINCIPLES

The minimum principle follows directly from eqn (28). Since the integrals $J(\boldsymbol{\epsilon}, \boldsymbol{\sigma}^1)$, $I^*(\boldsymbol{\sigma})$, $L(\omega, \tau^1)$ are nonnegative (see Section 4) the energy increment ΔE is bounded from above by first two integrals appearing in (28). The integrals $J(\boldsymbol{\epsilon}, \boldsymbol{\sigma}^1)$, $I^*(\boldsymbol{\sigma})$, $L(\omega, \tau^1)$ vanish for $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^1$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0$.

The above properties are sufficient to establish the **minimum principle:**
The function

$$F_u(\boldsymbol{\epsilon}, \boldsymbol{\sigma}) = \int_{V_1} [W(\boldsymbol{\epsilon}, \mathbf{x}) - \boldsymbol{\epsilon} \cdot \boldsymbol{\sigma} + W^*(\boldsymbol{\sigma}, \mathbf{x})]\, \mathrm{d}V + \int_C G(\omega)\, \mathrm{d}C \tag{34}$$

defined for all functions $\boldsymbol{\epsilon}(\mathbf{x})$ from the set K_1 and all functions $\boldsymbol{\sigma}(\mathbf{x})$ from the set S_0 attains the minimum equal to ΔE at $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^1$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0$. Here the scalar function $\omega(\mathbf{x})$ representing relative displacement of the crack surfaces is determined (5) *by the strain function $\boldsymbol{\epsilon}(\mathbf{x})$.*

In an analogous way, making use of eqn (33) and the properties of the integrals introduced in Section 4, we establish the **maximum principle:**
The function

$$F_\ell(\boldsymbol{\epsilon}, \boldsymbol{\sigma}) = -\int_{V_1} [W(\boldsymbol{\epsilon}, \mathbf{x}) - \boldsymbol{\epsilon} \cdot \boldsymbol{\sigma} + W^*(\boldsymbol{\sigma}, \mathbf{x})]\, \mathrm{d}V - \int_C G^*(\tau)\, \mathrm{d}C \tag{35}$$

defined for all functions $\boldsymbol{\epsilon}(\mathbf{x})$ from the set K_0 and all functions $\boldsymbol{\sigma}(\mathbf{x})$ from the set S_1 attains the maximum equal to ΔE at $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^1$. Here the scalar function $\tau(\mathbf{x})$ representing the normal tractions on the crack surfaces is determined (8) *by the stress function $\boldsymbol{\sigma}(\mathbf{x})$.*

6. CONCLUDING REMARKS

The extremum principles established in Section 5 are formulated for the linear sets $K_1 \times S_0$ (minimum principle) and $K_0 \times S_1$ (maximum principle) of the arguments $\boldsymbol{\epsilon}(\mathbf{x})$, $\boldsymbol{\sigma}(\mathbf{x})$. One can modify the formulation of the extremum principles by introducing the convex set K_c of all *kinematically admissible* strain functions $\boldsymbol{\epsilon}(\mathbf{x})$ from K_1

$$K_c = \left\{\boldsymbol{\epsilon}(\mathbf{x}): \boldsymbol{\epsilon}(\mathbf{x}) \in K_1, \int_C G(\omega)\, \mathrm{d}C = 0\right\} \tag{36}$$

and the convex set S_c of all *statically admissible* stress functions $\boldsymbol{\sigma}(\mathbf{x})$ from S_1

$$S_c = \left\{\boldsymbol{\sigma}(\mathbf{x}): \boldsymbol{\sigma}(\mathbf{x}) \in S_1, \int_C G^*(\tau)\, \mathrm{d}C = 0\right\}. \tag{37}$$

It follows from the definition (7) of the function $G(\omega)$ that $F_u(\boldsymbol{\epsilon}, \boldsymbol{\sigma}) = +\infty$ for arbitrary $\boldsymbol{\epsilon}(\mathbf{x}) \notin K_c$ and $\boldsymbol{\sigma}(\mathbf{x}) \in S_0$ and that $F_\ell(\boldsymbol{\epsilon}, \boldsymbol{\sigma}) = -\infty$ for arbitrary $\boldsymbol{\epsilon}(\mathbf{x}) \in K_0$ and $\boldsymbol{\sigma}(\mathbf{x}) \notin S_c$. Hence we can reformulate the extremum principles using the convex sets of the arguments:

Minimum principle.

The function

$$F_u(\boldsymbol{\epsilon}, \boldsymbol{\sigma}) = \int_{V_1} [W(\boldsymbol{\epsilon}, \mathbf{x}) - \boldsymbol{\epsilon} \cdot \boldsymbol{\sigma} + W^*(\boldsymbol{\sigma}, \mathbf{x})]\, \mathrm{d}V \tag{38}$$

defined for all kinematically admissible strain functions $\boldsymbol{\epsilon}(\mathbf{x})$ *in body* 1 ($\boldsymbol{\epsilon}(\mathbf{x}) \in K_c$) *and for all statically admissible stress functions* $\boldsymbol{\sigma}(\mathbf{x})$ *in body* 0 ($\boldsymbol{\sigma}(\mathbf{x}) \in S_0$) *attains the minimum equal to* ΔE *at* $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^1$ *and* $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0$.

Maximum principle.

The function

$$F_\ell(\boldsymbol{\epsilon}, \boldsymbol{\sigma}) = -\int_{V_1} [W(\boldsymbol{\epsilon}, \mathbf{x}) - \boldsymbol{\epsilon} \cdot \boldsymbol{\sigma} + W^*(\boldsymbol{\sigma}, \mathbf{x})]\, \mathrm{d}V \tag{39}$$

defined for all kinematically admissible strain functions $\boldsymbol{\epsilon}(\mathbf{x})$ *in body* 0 ($\boldsymbol{\epsilon}(\mathbf{x}) \in K_0$) *and for all statically admissible stress functions* $\boldsymbol{\sigma}(\mathbf{x})$ *in body* 1 ($\boldsymbol{\sigma}(\mathbf{x}) \in S_c$) *attains the maximum equal to* ΔE *at* $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0$ *and* $\boldsymbol{\sigma} = \boldsymbol{\sigma}^1$.

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APPENDIX

Application of convex analysis to the crack problem

The objective of this appendix is to present a brief introduction to the mathematical concepts used in this article for the reader who is not familiar with convex analysis. The precise definitions and full description of the mathematical objects considered by convex analysis can be found in [4, 5].

The relation between the force $\mathbf{T}$ and the relative displacement $\mathbf{u}^+ - \mathbf{u}^-$ of the crack boundary [see eqns (5–7)] is expressed in terms of the *subdifferential* of the convex function $G(\omega)$ which represents the frictionless bounds. It should be noted that function G determines the force $\mathbf{T}$ for the entire range of the displacement which consists of three intervals: the open crack ($\omega < 0$) where $\mathbf{T} = \mathbf{0}$, the closed crack ($\omega = 0$) where $\mathbf{T}$ is normal to the boundary and assumes arbitrary nonnegative value and physically impossible "self-overlapping crack" ($\omega > 0$) where the set of corresponding vectors $\mathbf{T}$ is empty.

The subdifferential of the function appearing in relation (6) is a natural generalization of the gradient for convex function which is not necessarily differentiable or even continuous. Indeed, if the scalar function f defined for all vectors $\mathbf{x}$ from the space X is convex and the value of f at $\mathbf{x}_0$ is finite then one can always construct the supporting plane P at the point $\mathbf{x}_0$ (see Fig. 2) where the vector $\mathbf{g}$ from the dual space X^* is called the *subgradient* of function f at $\mathbf{x}_0$.

By the definition the subgradient g satisfies the inequality

$$f(\mathbf{x}) - f(\mathbf{x}_0) - (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{g} \geq 0 \tag{A1}$$

for arbitrary $\mathbf{x}$ from X. Hence for every $\mathbf{x}_0$ one can construct the set of all subgradients which is called the subdifferential of function f at $\mathbf{x}_0$ and denoted by $\partial f(\mathbf{x}_0)$.

If convex function f is finite at $\mathbf{x}_0$ then the subdifferential contains either one subgradient (in this case f is differentiable at $\mathbf{x}_0$ and the subgradient is identified as the gradient) or an infinite number of distinct subgradients. Otherwise the subdifferential of function f at $\mathbf{x}_0$ is empty.

The function $G(\omega)$ defined by (7) is a simple example of convex lower-semicontinuous function. For arbitrary $\omega < 0$ (open crack) the subdifferential contains one subgradient (or gradient) $\tau = 0$. At the point of discontinuity $\omega = 0$ (closed crack) the subdifferential contains infinite number of subgradients $\tau \geq 0$. For arbitrary $\omega > 0$ (self-overlapping crack) the subdifferential is empty, i.e. there is no force corresponding to such displacement.

The concept of subdifferential is also used to formulate the relation between the strain and stress for a generalized elastic body. Such generalization allows one to use the resulting extremum principles for wider class of problems including for example an elastic-partially rigid body. Indeed, the rigid part of the body can be characterized by the free-energy function W vanishing for $\boldsymbol{\epsilon} = \mathbf{0}$ and equal to $+\infty$ for the remaining strain tensors. Then the subdifferential of function W at $\boldsymbol{\epsilon} = \mathbf{0}$ contains infinite number of stress tensors $\boldsymbol{\sigma}$ (arbitrary stress tensor corresponds to the strain $\boldsymbol{\epsilon} = \mathbf{0}$) and the subdifferential is empty for all remaining strain tensors (any deformation of the rigid body is not admissible).

It should be noted that the generalization of the elastic body does not make the derivation of the extremum principles more difficult. It follows from the fact that the differentiability of the free energy function is never used in the derivation of the extremum principle while the convexity of this function plays an essential role.

The derivation of the extremum principles is based on the properties of the *polar function*. The function $f^*(\mathbf{g})$ polar[5] (or *conjugate*[4]) to the function $f(\mathbf{x})$ is defined as

$$f^*(\mathbf{g}) = \sup_{\mathbf{x} \in X} [\mathbf{x} \cdot \mathbf{g} - f(\mathbf{x})] \tag{A.2}$$

i.e. the value of the polar function for given vector $\mathbf{g}$ is equal to the supreme value of the function $\mathbf{x} \cdot \mathbf{g} - f(\mathbf{x})$ in the vector space X.

In the particular case of the elastic body the function $W^*(\boldsymbol{\sigma})$ polar to the free energy function $W(\boldsymbol{\epsilon})$ represents the complementary energy of the body.

In the derivation of the extremum principle we make use of the inequality

$$f(\mathbf{x}) - \mathbf{x} \cdot \mathbf{g} + f^*(\mathbf{g}) \geq 0 \tag{A3}$$

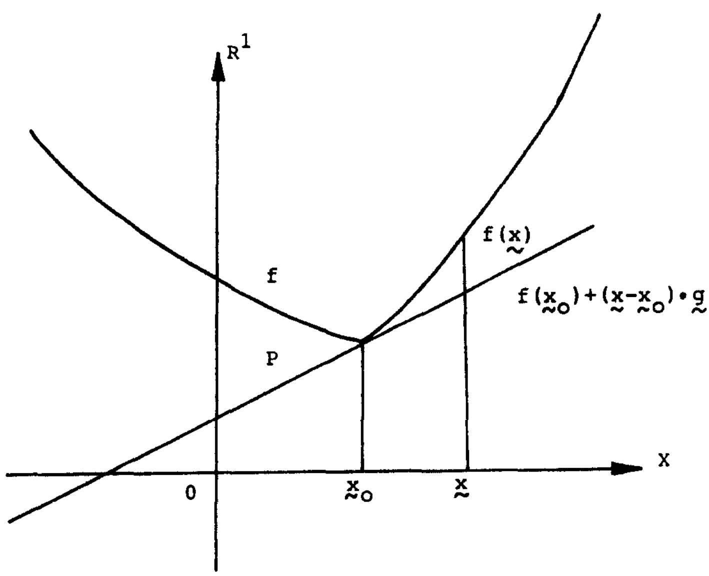

Fig. 2. Convex function f and plane P supporting f at $\mathbf{x}_0$.

which is true for arbitrary $\mathbf{x}$ and $\mathbf{g}$ and follows directly from the definition (A2) of the polar function. Indeed, for arbitrary $\mathbf{g}$ and $\mathbf{x}$ we have

$$f^*(\mathbf{g}) \geq \mathbf{x} \cdot \mathbf{g} - f(\mathbf{x}) \tag{A4}$$

as $f^*(\mathbf{g})$ is defined as the supreme value of the function on the right side of the inequality.

It should be noted that the entire derivation of the extremum principles is reduced to the application of two inequalities: (A1) and (A3).

The functions F_u and F_l appearing in the primary formulation of the extremum principles are defined for the full range of arguments which includes both physically admissible and physically inadmisible states. Physically inadmissible sets of arguments are here characterized by the infinite value of the function (numbers $+\infty$ and $-\infty$ are authorized in convex analysis). This property makes it possible to limit the range of search for extremum value to the set of physically admissible arguments (see Section 6).